# Mechanisms for Phase Transitions in the Multifractal Analysis of Invariant Densities of Correlated Random Maps *

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Multifractal properties of the invariant densities of correlated random maps are analyzed. It is proven that within the thermodynamical formalism phase transitions for finite correlations may be due to transients. For systems with infinite correlations we show analytically that phase transitions can occur as a consequence of localization–delocalization transitions of relevant eigenfunctions.

*Key words:* Fractals, Dynamical systems, Phase transitions, Stochastic automata, Random walk.

## 1. Introduction

Random maps, widely known as iterated function systems (IFS) [1], arise in many physical systems. Examples are Ising chains in random fields [2], variants of the Anderson model [3], prediction problems [4], and learning in neural nets [5]. A second motivation for the study of such systems comes from nonlinear dynamics in so far as one directly defines the symbolic dynamics, and thus circumvents the often difficult task of finding a good symbolic representation.

## 2. Correlated Random Maps, Multifractals, and their Generation with Stochastic Automata

A text book example of the connection between random maps and multifractals is the binomial multiplicative measure [6]. This multifractal measure may be viewed as the (integrated) invariant density of an associated IFS, where with probability $p$ one iterates with map $f_0(x_t)$ i.e. $x_{t+1} = f_0(x_t) = x_t/2$, and with probability $1-p$ one iterates with $f_1(x_t) = x_t/2 + 1/2$. The generated multifractal is fully characterized by the allowed sequences of maps, for convenience coded as symbol sequence $S_1 S_2 \dots S_N$ with $S_i \in \{0, 1\}$, and the corresponding probabilities $p(S_1 S_2 \dots S_N)$, $N$ arbitrary. For the binomial measure all possible symbol sequences are also allowed sequences, and the associated probability measure is simply $p^s (1-p)^{N-s}$ with $s$ being the number of zeros in a symbol string of length $N$. This probability measure expresses the fact that in this special case the maps $f_0$ and $f_1$ are applied randomly in an uncorrelated manner. In the following we consider cases where the probabilities $p(S_1 S_2 \dots S_N)$ are ruled by Markov processes with memory of length $M$ (representable as finite automaton), including the limiting cases of $M \to \infty$ (infinite automaton).

In a multifractal analysis one considers "partition sums" of the form $Z_q(N) = \sum p^q (S_1 S_2 \dots S_N)$, where the summation is over all allowed symbol sequences of length $N$. In the limit $N \to \infty$ they grow typically exponentially with $N$. i.e. $Z_q(N) \sim \exp[N \varkappa(q)]$, where $\varkappa(q)$ is related to the generalized entropies $K_q$ by $\varkappa(q) = (1-q) K_q$ [7]. In the case of equal contraction ratios $a$ of the maps (for the multiplicative binomial measure $a = 1/2$) $\tau(q)$, the Legendre transform of the $f(\alpha)$ spectrum [8], is proportional to $\varkappa(q)$ [9]. In the following we consider only this univariate case and therefore may for simplicity set $\varkappa(q) = -\tau(q)$ (corresponding to $a = 1/e$). With this convention $\tau(q)$ of the binomial measure simply reads $\tau(q) = -\ln[p^q + (1-p)^q]$. A general finite stochastic automaton is defined [10] by the transition matrices $P(S)$, with $S$ an element of the alphabet $\Sigma$. A matrix element $p_{kl}(S)$, $k, l = 1, \dots, n$ is the probability for making a transition from node $k$ to node $l$ of the automaton consisting of $n$ nodes, and thereby emitting the symbol $S$. The transition matrix $P = \sum_S P(S)$ is a stochastic matrix. The probability for generating a symbol sequence $S_1 S_2 \dots S_N$ is given by $p(S_1 S_2 \dots S_N) = \pi P(S_1) P(S_2) \dots P(S_N) \eta$, where $\pi$ is the $n$-dimensional vector of the initial probability distribution over the nodes, and $\eta = (1, 1, \dots, 1)$ provides a summation over all nodes. In this general case

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$Z_q(N)$, $\tau(q)$, etc. cannot be calculated analytically for all $q$. For Markov automata, however, a reduction to an eigenvalue problem is possible: In this case one has a one-to-one correspondence between path and symbol sequence, and the sum over all symbol sequences of length $N$ can be evaluated. One obtains $Z_q(N) = \pi P_q^N \eta$ with $[P_q]_{kl} = \sum_S [p_{kl}(S)]^q$. For $N \to \infty$ the maximal eigenvalue $\lambda_{\max}(q)$ of $P_q$ dominates, resulting in $\tau(q) = -\ln[\lambda_{\max}(q)]$. This situation is quite analoguous to the calculation of the topological pressure from a Markov partition for hyperbolic dynamical systems (see e.g. [11]).

## 3. Phase Transitions

In analogy to statistical thermodynamics, phase transitions are defined as non-analycities in the $q$-dependence of $\tau(q)$ [12]. Since the matrix $P_q$ is non-negative and finite for finite $M$, the Perron-Frobenius theorem [13] applies. This means that for an ergodic, non-cyclic Markov process the usual transition matrix $P_{q=1}$, and thus also $P_q$, is irreducible. Therefore $\lambda_{\max}(q)$ is simple and also the largest in absolute value. It is known that its dependence on $q$ is analytic and therefore for finite $q$ no phase transitions are possible. This is in full analogy to the absence of phase transitions in $1-d$ Ising spin chains with short-range interactions, where these arguments apply to the transfer matrix. Thus one is left with two possibilities to avoid the consequences of Perron-Frobenius and to find phase transitions: Firstly, non-ergodic Markov chains corresponding to non-ergodic finite automata (ergodic, cyclic processes are not treated here), and secondly, infinite automata or processes with infinite memory.

### *a) Non-Ergodic Finite Automata*

In the following we briefly show that in this case, which physically corresponds to transient behaviour, one can get phase transitions which are typically of first order. In contrast to the ergodic case, $P_q$ is now reducible. Thus [13] there exists a renumbering of the states such that the matrix $P_q$ becomes a generalized triangular matrix consisting of two square matrizes $A_q$ and $B_q$ ($n_A \times n_A$ and $n_B \times n_B$, with $n_A + n_B = n$) as diagonal elements, and a rectangular matrix $R_q$ ($n_A \times n_B$) as upper off-diagonal element. The lower off-diagonal element is a rectangular null matrix. Physically, $A_q$ describes transitions within the transient part, $B_q$ transitions within the ergodic part, and $R_q$ transitions from the transient to the ergodic part of the Markov automaton. A consequence of this form is that the eigenvalue problem decomposes into two independent problems since $\det(P_q - \lambda E) = \det(A_q - \lambda E_A) \cdot \det(B_q - \lambda E_B)$. Here $E$, $E_A$, and $E_B$ are $n$-, $n_A$-, and $n_B$-dimensional unit matrices, respectively. Thus a crossing of the maximal eigenvalues of $A_q$ and $B_q$ becomes possible, with the result that $\lambda_{\max}(q)$ is non-analytic near the crossing at $q_c$. In the following we prove that such mechanisms for phase transitions are realized very easily: Assume that the states in the transient and in the ergodic part of the automaton are each fully connected, i.e. $A_q$ and $B_q$ contain only non-zero entries. Then it follows that all elements of $A_{q=0}$ and $B_{q=0}$ are equal to one. Therefore their maximal eigenvalues are $n_A$ and $n_B$. On the other hand, the matrix $B_{q=1}$ is stochastic since $P_{q=1}$ is stochastic, and $A_{q=1}$ is substochastic since $R_q$ contains non-zero elements. This means that $\lambda_{\max}(B_{q=1}) = 1$, whereas $\lambda_{\max}(A_{q=1}) < 1$. For $n_A > n_B$ we thus get $\lambda_{\max}(A_{q=1}) < \lambda_{\max}(B_{q=1})$ and $\lambda_{\max}(A_{q=0}) > \lambda_{\max}(B_{q=0})$. And, since both maximal eigenvalues are continuous functions of $q$, they have to cross at some value $q_c$ with $0 < q_c < 1$. Further, two curves typically intersect transversally, which means that the derivative of $\lambda_{\max}(q)$ and $\tau(q)$ is discontinuous at $q_c$. Thus the obtained phase transition is generically of first order. We remark that based on the known bounds for the maximal eigenvalues of non-negative matrices [13], and with a similar reasoning as above, one can formulate more general sufficient conditions for the existence (and also for the non-existence) of such phase transitions.

### *b) Infinite Automata*

In order to understand possible mechanisms for phase transitions in iterated function systems and fractals with infinite correlations in their hierarchy, we consider in the following one of the simplest infinite automata which generates such correlations. It consists of a semi-infinite chain of states or nodes $A_i$, $i = 1, 2, 3, \ldots, \infty$ with transitions only between the nearest neighbour nodes. Specifically, a symbol "1" is generated during transitions from states $A_k$ to $A_{k+1}$ with probability $p$ (for $k \neq 1$), and with probability $1-p$ a symbol "0" is emitted in transitions from $A_k$ to

$A_{k-1}$, $k \geq 2$. At node $A_1$ with certainty ($p = 1$) a transition with symbol "1" is made to node $A_2$. Without the generation of symbols the defined automaton is simply a model for a discrete random walk with a reflecting barrier. The latter model is fully solvable, i.e. eigenvalues and the corresponding left and right eigenvectors of the transition matrix $P$ are known [14]. As a consequence, the mutifractal properties of the generated fractal can also be calculated exactly.

Let us assume for simplicity that the initial probability distribution is concentrated on the state $A_1$. With the knowledge of this initial state we have again a one-to-one correspondence between symbol sequences and state sequences. Thus, as in the finite memory case there is only one product of transition probabilities which for a given symbol combination of length $N$ contributes to the partition sum $Z_q(N)$. This again allows the summation over symbol sequences and one obtains $Z_q(N) = \pi P_q^N \eta_N$, where $P_q$ is now an infinite matrix with elements $[P_q]_{12} = 1$, $[P_q]_{k,k+1} = p^q$, $[P_q]_{k,k-1} = (1-p)^q$, for $k \geq 2$, and zero otherwise. $P_{q=1}$ is just the one-step propagator $P$ of the random walk. The infinite vectors $\pi = (1, 0, 0, \ldots)$ and $\eta_N = (1, 1, \ldots, 1, 0, 0, \ldots)$, with the first $N+1$ entries equal to one, reflect the above initial condition and the fact that only up to $N+1$ nodes can be reached by emitting $N$ symbols. Unfortunately, due to the $N$-dependence of $\eta_N$ one cannot conclude, as in the Markovian case, that simply $\lambda_{\max}(q)$, the maximal eigenvalue of $P_q$, determines the asymptotic growth of $Z_q(N)$. Effectively one deals with a matrix which grows with $N$. In extending Kac's method [14] to $q \neq 1$, we calculated eigenvalues and eigenvectors of $P_q$, inserted the spectral decomposition of this matrix into the expression for $Z_q(N)$, and with an asymptotic analysis we obtained the growth exponent $\tau$. It turns out that depending on $q$ and the probability $p$, one gets 3 different forms for $\tau(q)$:

$$\tau_d(q) = -\frac{1}{2} \ln\left(\frac{(1-p)^q}{1-p^q}\right),$$

$$\tau_b(q) = -\ln(2) - \frac{q}{2}\ln(p(1-p)), \tag{1}$$

$$\tau_f(q) = -\ln(p^q + (1-p)^q),$$

where the indices d, b, and f stand for discrete, band edge, and free random walk, respectively. In Fig. 1 the resulting $\tau(q)$ for $p = 0.25$ is depicted. For $-\infty < q \leq 0$ the form $\tau_f(q)$ is valid, in the interval $0 \leq q \leq q^* = \ln(1/2)/\ln(p)$ one finds $\tau(q) = \tau_b(q)$, and for $q \geq q^*$ we obtain $\tau(q) = \tau_d(q)$. At the critical points $q = 0$ and $q = q^*$ the function $\tau(q)$ varies continuously with a jump in the second derivative, and we therefore find two second order phase transitions. The transition at $q^*$ can be understood by a change in the nature of the spectrum of $P_q$. This is shown in Fig. 2 as function of $q$: For $q \leq q^*$ the spectrum is absolutely continuous, we have a band of eigenvalues. In the regime $q > q^*$ there exists in addition a discrete eigenvalue outside the band and therefore of maximal modulus. This is also found by a calculation of the density of states through a continued fraction expansion of the corresponding Green's function (see e.g. [15]). It turns out that in both cases $\lambda_{\max}$ of $P_q$ determines $\tau(q)$: In one case it is the band edge, in the other it is the discrete

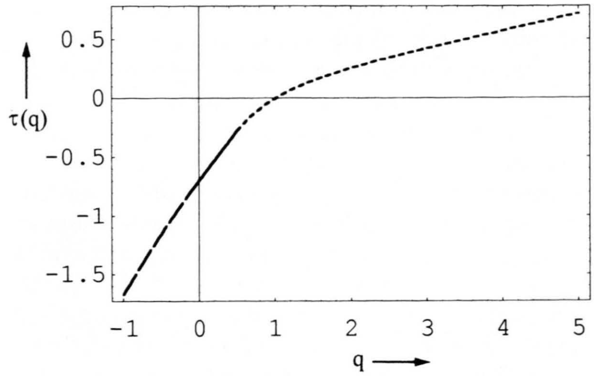


Fig. 1. $\tau(q)$ for the random walk model with $p = 1/4$: The domains of analyticity are separated by two second order phase transition points at $q = 0$ and $q^* = 0.5$, respectively. The "free" behaviour for $q \leq 0$ is the same as for the binomial multiplicative measure.

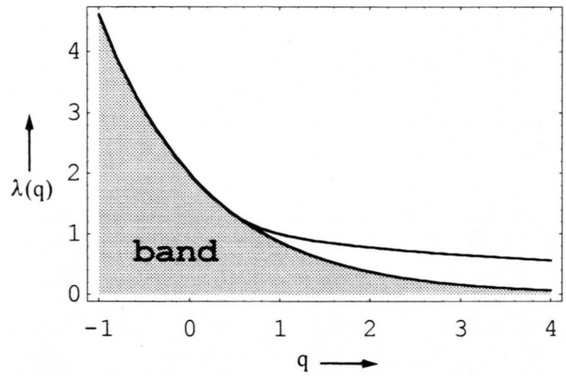


Fig. 2. The spectrum $\lambda(q)$ of the generalized propagator $P_q$. The merging of the discrete eigenvalue with the band at $q^* = 0.5$ causes a second order phase transition (see Figure 1).

eigenvalue, and the phase transition is due to the merging of both. The physical origin of this transition can be understood as follows: By a look at the matrix $P_q$ one recognizes that lowering $q$ for fixed $p$ has a similar effect as enhancing $p$ for fixed $q$. Fix $q$ at $q=1$, then for $p<1/2$ there exists near the origin $A_1$ an exponentially decaying stationary distribution in the random walk problem, the left eigenfunction of $P=P_{q=1}$ corresponding to the isolated eigenvalue $\lambda_{\max}=1$ (a surface state in the language of related solid state problems [15]). As $p$ is enhanced across $p=1/2$ the random walker drifts in a diffusive way to infinity, an equilibrium distribution no longer exists, and correspondingly the state with eigenvalue $\lambda_{\max}$, and also all other eigenfunctions (of the appropriately symmetrized $P_q$) are extended. It is this kind of transition which occurs by lowering $q$ across $q^*$, which is therefore properly called a localization–delocalization transition. The meaning of the localized state is best seen for $q\to\infty$, where the form of $\tau_d(q\to\infty)$ reflects that the most probable path is a cycle between states $A_1$ and $A_2$. The second transition at $q=0$ can no longer be understood solely in terms of the spectrum, since for $q<0$ the effect of $\eta_N$ becomes relevant. However, the fact that for $q<0$ the form of $\tau(q)$ is that of the free chain without barrier is intuitively clear: Negative $q$'s probe improbable paths in the automaton, and for $p<1/2$ these are provided by paths going to infinity which are not affected by the barrier. The case $p>1/2$ is slightly more complicated. We mention that, although the same equations (1) enter, one can also get first order phase transitions. This, together with the resulting "phase diagram" in the $p$–$q$ plane, will be published elsewhere [16].

## 4. Discussion

Similar results as in the last example are also found in bi-infinite models with one or more "impurities", i.e. where at one or several nodes the transition probabilities are changed [16]. This has interesting consequences in so far as many concepts from localization theory in solid state physics apply also to fractals and dynamical systems. The latter follows from the fact that simple dynamical systems can be found where such a symbolic dynamics applies [16]. These new mechanisms extend the present picture of phase transitions where only discrete eigenvalues of the generalized Frobenius-Perron operator enter [11, 17].

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[1] M. F. Barnsley and S. Demko, Proc. Roy. Soc. A **399,** 243 (1985).
[2] R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **50,** 1494 (1983).
[3] F. Martinelli and E. Scoppola, J. Stat. Phys. **50,** 1021 (1988).
[4] D. Zamballa and P. Grassberger, Complex Systems **2,** 269 (1988).
[5] R. Radons, H.-G. Schuster, and D. Werner, Phys. Lett. A **174,** 293 (1993).
[6] J. Feder, Fractals, Plenum Press, New York 1988.
[7] H. G. E. Hentschel and I. Procaccia, Physica D **8,** 435 (1983).
[8] T. C. Halsey et al., Phys. Rev. A **33,** 1141 (1986).
[9] T. Tél, Z. Naturforsch. **43a,** 1154 (1988).
[10] A. Paz, Introduction to Probabilistic Automata, Academic Press, New York 1971.
[11] C. Beck and F. Schlögl, Thermodynamics of Chaotic Systems, Cambridge University Press, Cambridge 1993.
[12] D. Katzen and I. Procaccia, Phys. Rev. Lett. **58,** 1169 (1987), and Refs. therein.
[13] F. R. Gantmacher, Matrizentheorie, Springer-Verlag, Berlin 1986.
[14] M. Kac, Amer. Math. Monthly **54,** 369 (1947).
[15] R. Haydock, Solid State Physics **35,** 215 (1980).
[16] G. Radons, to be published.
[17] M. J. Feigenbaum, I. Procaccia, and T. Tél, Phys. Rev. A **39,** 5359 (1989).